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THE RECOVERY OF COMPREHENSIBLE MATHEMATICS

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Abstract. *The main objective is to get over the gap that exists between mathematics and common people, especially grown up people. Apart mathematical details, the problem lies in a good choice of notices (curiosity) and nice problems (play). Some historical notes about great mathematicians are presented and discussed, with explicit reference to the cases when the boundary between Nobel prize and mathematics was broken. Favourable fields are probability and operations research. Since probability tends to an excess of theory, operations research seemed to be a good choice. The Fields Medal, a kind of Nobel prize for Mathematics, was also considered, since in 2018 it was achieved by the Italian mathematician Figalli, former student of Scuola Normale Superiore di Pisa. He started from an important field in the frame of Operations Research, namely Optimal Transport. This sector allows to summarize a very nice procedure for its solution, non at all obvious to be trasferred to the computer. Since mathematics is forgotten in the course of life, except for those few parts of current use, to bring the adult back into the interest of mathematics, topics related to everyday life should be presented. Operations research, and especially network optimization, provide significant but pleasing problems.*
Keywords: *Fields Medal, Optimal Mass Transport, Linear Programming, Stepping Stones, Duality*

Introduction

Mathematics is forgotten in the course of life, except for those few parts of current use. To bring the adult back to the interest in mathematics, topics related to everyday life and (possibly) free from school exercises should be presented. However, curiosity for facts linked to the scientific actuality must also be satisfied.

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The most sensational events in the world of science are the Nobel prizes. Everybody knows that the prize is awarded every year to scholars and to men of culture of the highest fame. In particular, for scientists there are prizes for physics, chemistry, medicine, more recently economics. The prize that is missing in science is that for mathematics, and therefore only rarely one of its scholars reaches, under false label, this goal. We will show later some examples between economics and mathematics.

There is also a world prize for maths. It is conferred only once every four years, on the occasion of the International Congress of Mathematicians. This is the Fields Medal, which is reserved for researchers under the age of 40, based on the fact that the maximum creativity of mathematicians, for consolidated experience, is achieved in youthful years. However, there is a risk that an important discovery will not win while its author is young, and there is no possibility of a change of mind. In the Nobel Prize, on the contrary, there were frequent cases of recovery that made the winner say "When I achieved the result, thirty years ago, I thought I deserved the Nobel Prize, ... now it has arrived".

The winners of the great scientific prizes usually come from prestigious research schools, and in the next section we will cite as an example the Scuola Normale Superiore di Pisa, where Alessio Figalli, Fields Medal 2018, was a pupil. We will also mention some of its eminent mathematicians.

As for a pleasant exercise, the choice will fall on the fashionable subject studied by the latest Fields Medal: optimal mass transport. The possibility of connecting an example that can be understood with a scientific result of world-wide value is an absolute rarity, and usually occurs only in the encounter between applicative problems and non-usual mathematical foundations. The economy, the social sciences, the non-deterministic biological and physical models, are lands of conquest more or less fortunate. In our example we will enter a field where it is possible to explain to the layman not only the result but also the technique with which the problem can be faced to obtain concrete results. The methodology section has this objective. The result section shows how they are interpreted to derive optimal solutions, and the conclusions suggest the critical analysis that must be used in the economic applications of mathematical procedures. In this way the reader will catch a glimpse of the road that led Figalli to his victory in the Fields Medal.

From History to Literature Review

The land of conquest of mathematicians is the economy, which to mention the most ancient cases, sees the victory of Vassily Leontieff (1973)[1], creator of the macroeconomic system of input-output matrices, of Leonid Kantorovich (1975)[2], who since the forties reopened the topic of mass transport, of the inventor of artificial intelligence Herbert A. Simon (1978) and that of the great game theory expert John F. Nash (1994).

The Scuola Normale Superiore of Pisa had among its students two Nobel Prizes for Physics (Enrico Fermi and Carlo Rubbia), and a mathematician for the Fields Medal, Figalli. It is one of the most prestigious scientific universities. For mathematics we must remember at least four distinguished names: Ennio De Giorgi (1928-1996), professor of analysis for 35 years; Enrico Bombieri, born in 1940, professor in the seventies, Fields Medal in 1974; Luigi Ambrosio (born in 1963) student and then fellow of the Accademia dei Lincei; Alessio Figalli (born in 1984), student and then graduate student, Fields Medal in 2018.

The name of De Giorgi crosses with that of Nash for the titanic challenge that engaged them between 1955 and 1958, aimed at solving the nineteenth problem of Hilbert (the regularity of the solutions of elliptic problems with discontinuous coefficients). De Giorgi came first (De Giorgi, 1957) starting, among other things, from sophisticated inequalities of isoperimetric type (generalized in De Giorgi, 1958), while Nash reached the goal in 1958 (Nash 1958) using the properties of the heat equation. Due to the concomitance of the results neither of them received the deserved Fields Medal. The story of those years is narrated with rich testimony by Parlangeli in his book on the life of De Giorgi (Parlangeli, 2015/2019, pages 57-67). In Piccinini's book on De Giorgi (Piccinini, 2016), the manuscript text of De Giorgi's demonstration is included in the appendix, as well as some comments on its contents (pag. 128-130). A broad discussion on the subject is found in the note by Piccinini-Lepellere (Piccinini & Lepellere, 2018), where the important final role of Moser (Moser, 1960) in the simplification of the proof is also specified.

In general, Fields Medal winners work in very sophisticated and complicated areas of mathematics, where only a few high-level and highly specialized colleagues can understand the extent of the findings. A philosopher would talk about esotericism. The exception is sometimes given by the theory of numbers, where the comprehensibility of some results, however, should not

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[1] The Input.Output matrix theory has led to the solution of various mathematical problems concerning the hierarchical order of economics, discussed in depth by some of the authors in (Chang, Piccinini, Iseppi, & Lepellere, 2013)

[2] He won with Tjalling C. Koopmans, who in those same years contributed to linear programming using the simplex method together with Dantzig.

make us think of ease of problems (falsely exoteric lessons). This was precisely the sector in which Enrico Bombieri won, in 1974 in Vancouver. But in those same years Bombieri participated enthusiastically and with refined authority also in the research of De Giorgi on the surfaces of the minimum area, a natural continuation of the research on the nineteenth problem of Hilbert (Bombieri, De Giorgi, & Giusti, 1969 and Bombieri, De Giorgi, & Miranda, 1969). At the *Lectio Magistralis* in Vancouver he surprised all those present, especially talking of his works with De Giorgi instead of his researches in number theory that had already given him international fame. In the opinion of who was present it seemed the first revenge of De Giorgi, while Nash had still to wait.

A very brilliant and original student of De Giorgi is Ambrosio, who took at the Scuola Normale the chair that was formerly his. He again touched the optimal mass transport sector, which had remained in the shadows for many years, being almost considered pure operationsl research. Here he has obtained important results, also connecting to problems dear to the Master, concerning the surfaces of minimum area and thin obstacles. Ambrosio in the book for the tenth anniversary of the death of De Giorgi (Ambrosio, 2008), wrote a note inspired by the theory of geometric measure, continuing the approach of the book completed years before by Piccinini and Colombini (De Giorgi, Colombini, & Piccinini, 1972 and Piccinini, 1973). His teaching was the basis of the formation of Figalli, his graduate and then Ph.D. student.

Figalli was the first Italian mathematician to renew Bombieri's success in winning the Fields Medal, 44 years later. Given his young age he did not have time to know De Giorgi, but through his teacher he has drawn numerous stimuli. In particular we like to observe that he inherited the curiosity for all the scientific knowledge, which distinguished the Master. In addition, he has developed an exceptional capacity for collaboration even in sectors that at first sight are far from mathematics, such as ecology, meteorology, and sustainable economic development policies, as is evident in some recent applications of Figalli and its collaborators. (De Filippis & Figalli, 2014; Figalli, 2018).

Inspired by this eclectic, but also understandable and stimulating approach, we have chosen a classic theme of optimal mass transport, combining geometric intuition, computer techniques, combinatorial calculus with applications to economics and logistics. The possibility of operating hands-on can give the reader a further reason for satisfaction.

Methodology

The problem. We will expose the problem in its classical form, referring for the most modern (but less readable) forms to Ambrosio (Ambrosio, 2003) and

Villani (Villani, 2009). We will provide the reader with the tools used in the traditional methods of solution so that he can experience them personally.

The problem of optimal mass transport was originally proposed by Monge (Monge, 1781) and then studied during the Second World War for military purposes as evidenced by the pioneering work of Kantorovich (Kantorovich, 1939).

The original problem is the following: a good available in deposits is required by the destinations. In the M deposits O_i are found $p_1, ..., p_M$ units and in N destinations Dj are required $r_1, ..., r_N$ units. For each route from O_i to D_j there is a unit price $c_{i,j}$ for the transport of materials. With a little trick one can suppose that the total availability of the deposits is equal to the total demand T. The problem consists in choosing the origins and the destinations served by each one, which we indicate with $x_{i\,j}$ from O_i to D_j, so that the total cost

$$C = \sum_{i,j} c_{i,j}\, x_{i,j} \tag{1}$$

is minimal, continuing to satisfy the consistency conditions on the origins and destinations.

The problem belongs to linear programming and can be solved by the simplex method found by Dantzig and Koopmans (Dantzig, 1963). Unfortunately, this straightforward method requires a huge increase in the number of unknowns. Why is there a faster method? Even the amateur understands a logical thread in the fact that it is easy to construct a feasible solution. Of course, in general it is not the best solution and therefore needs to be improved.

The easiest way is called the "north-west corner". It starts from the first deposit and replenishes the first destination until either the availability of the origin or the request of the destination is exhausted. In the first case we proceed with the second origin, in the second case with the second destination and so on. In the last step the last row and the last column are exhausted together. All solutions that, like this one, have M+ N - 1 non-zero elements are called "basic solutions". Except for degenerate cases the optimal solution is a basic solution.

Originally when the calculation was largely manual, and when computers were not very powerful, it was useful to start from a basic solution that was already closer to the probable end result. The method of the northwest corner was not always bad because the problem was often geographic and the costs were proportional to the distance between storage and destination. If the origins are listed according to a reasonable geographical order and destinations as well, it can already provide a good basic solution.

There are more powerful methods of approximation that take into account transport costs from the beginning, and then accelerate the process using a preparatory work. Classic are the minimum methods per row or per column and the minimum per matrix, which are usually quoted in the classical books of operational research.

Tools: graphs and trees. Graphs and trees are elements of daily life that show a strong interconnection between combinatorics, geometry and information structure. Graphs represent the schematization of a network in which interconnections count, but not the actual design of their route (railways, subways). The method of representation using graphs is very old, so much that one of the most famous works is the Peutingerian tabula published by Miller (Miller, 1887). It represents the roads of the Roman empire, and the linear compression implemented to obtain the portability of the roll distorts all the proportions, but not the significant elements relating to the road nodes and the main localities that dot the streets. To remedy this, it is useful to explicitly write data about the nodes, such as the name, or the size of the city, but also data relating to the arcs that join the nodes, such as the length.

Among the many types of graph a particular relief, even conceptual, is assumed by trees. In this case there is a system of nodes, one of which forms the root of the tree, while the others are reached by a single arc coming from the lower levels. In this way all the nodes are reached and there are no cycles that come back on themselves. In this way every tree of N nodes (including the root) possesses exactly N-1 arcs. Figure 1a illustrates the example of a tree, in which, as in the genealogical trees, the root is represented at the top.

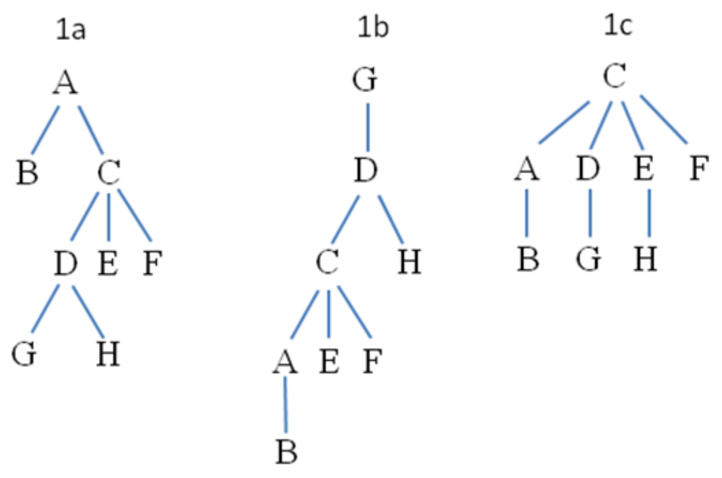


Figure 1 ***Equivalent trees with change of the root***

In a tree, any node can be taken as a root, and there are some invariants, first of all the number of arcs necessary to establish the connection between two chosen nodes. Figure 1b shows the tree 1a starting from one of its final branches, while the 1c shows it starting from an intermediate node.

On the computer efficacious representation of a tree is required. The visual image, comfortable for a man, does not allow further geometrical and algebraic elaborations. It is necessary to represent structural data without losing important information. The first systematic analyses were performed using the connected lists created by Simon and his collaborators in the pioneering study on artificial intelligence (Newell, Shaw, & Simon, 1957) as he recalls in his autobiography (Simon, 1991). Actually, connected lists became one of the pillars of information technology only with the work of Knuth (Knuth, 1968).

The representation of order 0 is the list of the nodes. Even dividing nodes into levels, there is the loss of information (who is the father?). The number of elements that come from each element of the higher level must be known at each level. Thus in the case of Figure 1a the list may be the following, and the patient reader can write the lists of 1b and 1c.

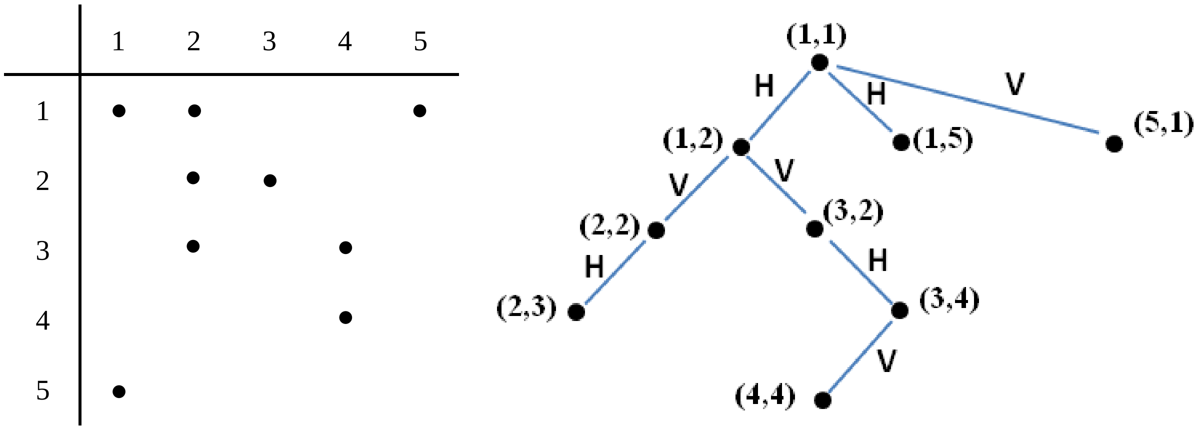
List 1a	
Level 0:	A;
Level 1:	B, C;
Level 2:	nil; D, E, F;
Level 3	G, H; nil; nil.

If the empty subsets (nil) were not reported, the origin of the nodes at levels 2 and 3 would no longer be known. This representation also makes it possible to distinguish the order in which descendants enter, and symbolically represents the succession visible in the figure.

Hierarchical representation of trees is common in managing folders on the computer. To go from one folder to another, it is necessary to go back up to the first common ancestor and then go back down along the other branch. This is the criterion that is also followed in the genealogical trees; there are cases in which the number of strings counts both upwards and downwards: in Italy the kinship goes up to the sixth degree, that is to say that for example a brother is a relative of second degree, an uncle is a relative of third, the cousins are fourth-degree relatives, the grandfather's brother is also fourth-degree, the son of a father's cousin is of the sixth grade and so on. The tree 1c represents the kinship related to the subject C, where at the first degree there is the father A and there are the sons D, E and F, while at the second degree there is the brother B and the two grandsons G and H. The degree of kinship remains unchanged also by changing the structure of the tree, as it only counts the number of branches. You can try to reconstruct the path from H to B in the three figures, always finding 4.

Orthogonal trees. A special type of tree is used in the quick procedures for resolving the discrete problem of the minimum cost assignments. The first step is to work on two matrices of M rows (origins) and N columns (destinations).

The former remains fixed and contains the unit costs of transferring from one source to a destination. The second matrix contains the assignments made according to availability and requests. The optimal solution is a basic solution that has exactly N + M -1 assignments, checking that each row and each column are covered. The procedure allows the system to be improved at each step by reducing the total cost until no improvement is possible. The assignments are characterized by a coordinate that expresses the row and one that expresses the column. The basic assignments are usually listed in row order from left to right.



2a ***Frame*** 2b ***Tree, root in (1,1)***

Figure 2 ***Rectangular complete Tree***

Figure 2a shows an example of data assignment and figure 2b shows the orthogonal tree. It starts from the first line and in it all the assignments are searched. The first assignment is assumed as the root of the tree and has the level 0. The other assignments are distinguished by level 1 and by the indicator H (horizontal). We then look for the assignments that are on the same vertical line, and they will also be marked by level 1 and by the indicator V (Vertical). No need to attribute a direction to the root. Then we proceed as in the normal analysis of any tree. In the second level there will be vertical assignments coming from H nodes, and will be indicated with the 2V level, or in horizontal assignments coming from V nodes, which will be indicated with level 2H. Why are you sure that no other descendants are horizontally starting from an H node? Try to do the analysis in Figure 2, starting from the root (1,1), and then, if you have worked correctly, you will find the answer.

Rectangular path. A rectangular path is that in which the direction is changed at each node. The path that connects two nodes according to what we saw in the section on the grammar of trees is already a rectangular path because each connector of a level with that of the upper level alternates the direction. The only exception can occur in the meeting node of two ascending paths. Here, in fact, both ascending branches can have the same direction. In this case the meeting node must be deleted.

We give two examples with the tree of Figure 2b

Liv 0 (1,1);
Liv 1 (1,2) H, (1,5) H, (5,1) V;
Liv 2 (2,2) V, (3,2) V; nil; nil;
Liv 3 (2,3) H; (3,4) H;
Liv 4 nil; (4,4) V.

For the example of figure 3a the orthogonal connection from (4,4) to (2,3) is needed. With the rules established for the exploration it results (4,4) V (3,4) H (3,2) V (1,2) and respectively (2,3) H (2,2) V (1,2). The welding knot is reached in the same direction and is therefore deleted. The resulting path is therefore (4,4), (3,4), (3,2), (2,2), (2,3).

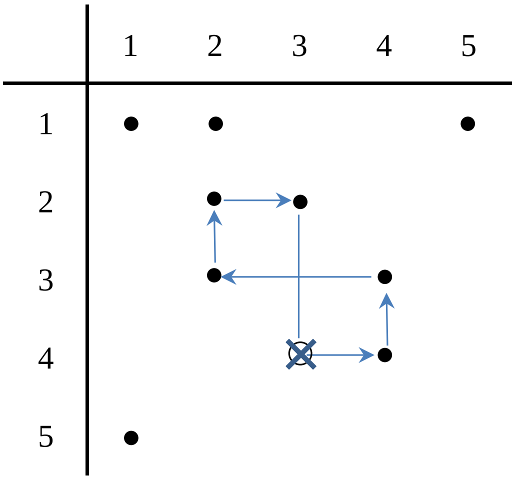


Figure 3a ***Closed path starting from (4,3)***

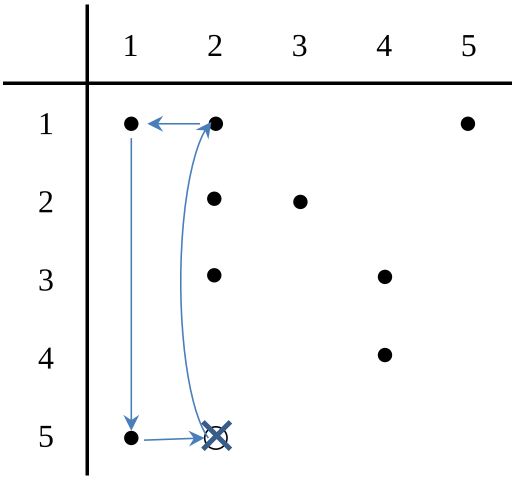


Figure 3b ***Closed path starting from (5,2)***

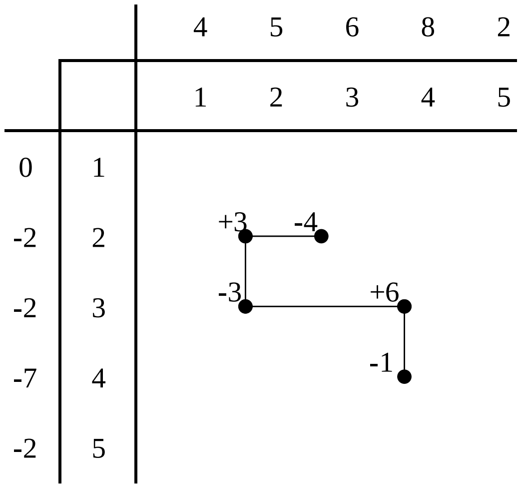
The *stepping stones* procedure is an extension of the previous one because a rectangular polygon (that is a closed rectangular path) must be constructed starting from a non-node point (v, h). The procedure involves finding in the vertical section starting from (v, h) either the node (y, h) belonging to the class H or the root, while in the horizontal section either the node (v, x) belonging to the class V, or the root. In the case of Figure 3b (v, h) = (5,2). It will be noted that in the search for the vertical connection there are three nodes (1,2) H, (2,2) V, (3,2) V. The only node classified H is to be chosen, namely (1,2). For the horizontal connection there is the node (5,1). The two paths (5,2) V (1,2) H (1.1) and in the other direction (5,2) H (5,1) V (1,1) will result. Since the welding knot is reached from different directions it remains, giving place to the path (5,2), (1,2), (1,1), (5,1), (5,2).

The value system. Before the assignment matrix there is the data matrix, where in each box are reported the unit costs from a source (row) to a destination (column). Once a basic allocation is made, it is useful to highlight in the data matrix only the committed boxes. In fig. 4 a possible set of costs associated with the matrix of fig. 2 has been reported. To simplify the subsequent calculations (MODI method) it is useful to characterize the rows and columns with a row indicator and a column indicator so that the cost of a committed box is the sum of the two indicators. In this way it appears that the indicators must satisfy a system of M + N -1 linear equations in M + N unknowns. Therefore one indicator of the root can be arbitrarily chosen and then the following are calculated in the order of exploration of the tree in fig. 2b. Each one is the difference between the cost of the current box and the last indicator found, as shown by the marginals of the matrix in fig.4.

		4	5	6	8	2
		1	2	3	4	5
0	1	4	5			2
-2	2		3	4		
-2	3		3		6	
-7	4				1	
-2	5	2				

Figure 4 **Frame 2 with quoted nodes. Value according tree of 2b [root in (1,1)]**

The rectangular polygon seen in the previous subsection plays a key role in improving the allocation structure. In fact, given an allocation, suppose to insert a new node. It will be inserted into a rectangular polygon with an even number of angles (2k). The new node is the reference point, indicated with 0, the following nodes in the path are numbered up to 2k-1. To keep the balance of the assignments the sum of each row and each column must remain equal. This is achieved by removing a unit in the odd nodes for each unit added to the new node, compensating by adding one unit in the even nodes. The result, given the construction of the indicators, is obtained by subtracting from the cost of the new unit (in the example (4,3)) the sum of the corresponding row and column indicators. Figure 5 shows the explicit sum of the costs of the removed nodes, which (changing the sign) is equal to the sum of the two indicators of the new unit, for a total -7 + 6 = -1. The indicator system corresponds substantially to the dual variables of the simplex method, as shown by the marginals of the matrix in fig.4.



-1+6-3+3-4=1
One unit added in (4,3) implies an increase of expense =1, plus the cost of (4,3)

Figure 5 ***Increase in (4,3) using the closed path (3a)***

If the cost of the new unit minus the variation is negative then the attribution to the new box decreases the total cost of the system. Since the nodes where the unit decrease occurs are the odd ones, the maximum amount that can be moved in the new box is equal to the minimum allocation of the odd boxes. The box that generated the minimum will disappear from the new base assignment, in the other nodes alternatively are added (even) and removed (odd) as many units as those of the new assignement[3]. Any other zeros remain assigned with value 0, so that the solution is always constituted by the M + N -1 assignments. If the minimum is 0, then in the new box the assignment is also 0. The problems where this situation occurs are considered degenerate, but they are usually resolved in the same way, provided a solution already examined is not chosen again. The problem is the same that occurs in the simplex procedure in the presence of degenerate cases. The procedure continues in the same way, each time looking for a new allocation where the difference in values is negative.

To verify that a solution is optimal (not necessarily unique) it is enough to build its set of indicators, and to verify that there are no boxes with a negative difference. The presence of null boxes implies the presence of more than one solution with the same minimum value. Remark that (fortunately) it is not necessary to explore every basic configuration, and usually the procedure ends very fast. Anyhow the number of the basic solutions is finite, and at each step a new solution is achieved, therefore the procedure has a finite stopping time.

[3] The possibility of varying the value depending on whether you add or remove a unit in the rectangular polygon shows that an optimal solution should not be able to undergo this process and therefore that there can be no more than M + N-1 allocations.

Analysis of the results

Before analyzing the way to eliminate multiple solutions it is necessary to explain the techniques that are used when there is no balance between the origins and the destinations. When resources are missing it is possible to add a fictitious deposit that covers the defect. By attributing the cost 0 to each of the destinations, one re-enters the canonical problem. In the case of oversupply, a fictitious destination is added that absorbs the surplus. Also in this case the basic choice is to impose the cost 0. The assignments of diversified costs for the fictitious cases allow to create a penalty system that privileges certain origins and destinations according to the needs of the system. These types of correction, and also other methods of estimating the variables, or splitting the final values into distinct blocks are generally explained in the operations research books, such as those of Ackoff-Sasieni (Ackoff & Sasieni, 1968) or Hillier-Lieberman (Hillier & Lieberman, 2005).

One way of eliminating the plurality of solutions in degenerate cases is that of data perturbation. The degenerate cases arise when the sum of the data of a part of the origins coincides with the sum of a part of the destinations. Then it is sufficient to perturb a little all the data of the origins, and add a fictitious column having the request equal to the sum. An example is $E = MIN / (2M + 2N)$, where MIN is the minimum of the source and destination data. The final result will be purified of the perturbation and will give the correct result without having the problem of avoiding cycles in the presence of multiple solutions. In particular if the data are all expressed by integers, then the solution is in turn made up of integers[4].

These observations allow *inter alia* to use the algorithm described above to solve the problem of optimal assignment[5]. The current procedure of using the simplex method is cumbersome, and therefore specific algorithms for this problem are preferred, but the mass transport method maintains a good efficiency. A good polynomial method is however the Hungarian method, described extensively by Lovasz-Plummer (Lovasz & Plummer, 1986) and well exemplified by Andreatta et al. (Andreatta, Mason, & Romanin, 1990).

For the multiplicity of optimal solutions we recall the classical example of the *Book shifting*. It requires that the N books placed in a shelf [0, N-1] be moved to the location [1, N]. If the order is not required, two extreme cases are possible: either all the books are moved by one unit (and in this case the order is preserved) or the books of [1, N-1] remain still and the first book migrates from

[4] When the problem is studied in linear programming, the matrix that characterizes it is totally unimodular, what guarantees that the solution in the presence of ineger data will also consist of integers.

[5] The students of one of the authors have implemented the algorithm described above and then it was actually used by colleagues of demography of the Faculty of Statistical Sciences of the University of Padua.

0 to N. If the cost of the transfer is equal to the length of the route both cases have the value N, in a case given by N * 1, in the other case given by (N-1)*0+1*N. At pag. 10 of Ambrosio's lectures (Ambrosio, 2009) the case of Cost = Distance is analyzed, but the case Cost = Distance * Distance (page 11) is very interesting. While the first case continues to give the value N = N*1*1, the second case (jump) gives a much greater value N*N = (N-1)*0+1*N*N. In this case the Brenier-Rachev-Knott-Smith theorem guarantees that the minimum solution (distributed shift) is unique (Rachev & Rueschendorf, 1989).

However, it is enough to make the model more realistic, by establishing that there is a set-up cost C for each operation, independent of the distance to be covered so that the presence of economies of scale can be detected, what could suggest the solution of the single movement, even if it is longer. This happens also in the elementary case of two positions.

In the case when the shift occurs into the following box we have the table 1a, while the case of the jump to the end is in the table 1b.

Table 1a ***Fundamental case of non-unicity with translation***

		C+1	2(C+1)		
		Cost 1	Cost 2	Ass 1	Ass 2
0	0	C+1	C+4	1	Index= -C+2
-(C+1)	1	0	C+1	0	1

Table 1b ***Fundamental case of non-unicity with jump***

		C+1	C+4		
		Cost 1	Cost 2	Ass 1	Ass 2
0	0	C+1	C+4	0	1
-(C+1)	1	0	C+1	1	Index= C-2

In the case of table 1a, if C = 2 the solution is optimal, but not unique. If C < 2 the solution is optimal and unique. If C > 2 the solution must be optimized by switching to table 1b. If C = 2 this solution is also optimal and therefore there is no uniqueness, otherwise the conclusions of the previous case are inverted so that C < 2 requires table 1a, while C > 2 gives the unique optimal solution.

The phenomenon is due to the lack of convexity of the cost function, so that the hypothesis of the Brenier-Rachev-Knott-Smith theorem is violated and therefore the uniqueness of the solution is not predictable. Economically the meaning is that if there are set-up costs at some point the economy of scale can arise (here C=2). For that value of indifference both solutions are acceptable.

Conclusions

While the resolutive techniques, such as the one shown above or that of the simplex serve to solve problems of large size, but not necessarily complex from the conceptual point of view, to understand the behavior of the solutions it is appropriate to resort to simple but significant examples. This is a concrete realization of Einstein's sentence "Things must be explained in the simplest way possible, but not simpler than that". The cases useful for understanding the phenomenon can be quite simple.

The reader can analyze the case of more than two nodes or the possibility to modify the cost allocation rules and draw the economic conclusions. Another interesting problem arises when the distance system is perturbed, as it happens during wars, earthquakes, floods. Resilience must fight against optimality. *Bounded rationality* (according Simon) is to be expected, but what is its price?

In operational research the boundaries between mathematics and its applications to the real world are very blurred and therefore are a source of reasoning combined with experience. The reader interested in a wide panorama of the optimal transport in the course of its historical evolution will be able to consult the book of Villani (Villani, 2009) and the last papers of Figalli (Figalli, 2018).

Summary

The paper starts with the description of the Fields Medal, equivalent of a Nobel prize for mathematicians. It is awarded every four years and is directed to mathematicians younger than 40. Generally the winners work in very complex sectors and only a few high-level colleagues can understand the extent of the findings. 2018 saw a new Italian victory after Bombieri in 1974. Alessio Figalli, formerly graduate student at Scuola Normale di Pisa, worked between mathematical analysis and operations research. This time even those who are not specialists can understand some of the problems faced, and can try to follow some solution procedures. Furthermore, applications to economics and logistics are easily seen. Figalli's research began with the problem of optimal mass transport, originally proposed by Monge and then solved during the Second World War for military purposes. An asset available in deposits is required by the destinations. In the m deposits O_i, we find $p_1, ..., p_m$ units and $r_1, ..., r_n$ units are required in the n destinations D_j. For each route from O_i to D_j there is a unit price c_{ij}, for the transport of materials. With a little trick one can suppose that the total availability of the deposits is equal to the total demand T. The problem consists in choosing the quantities x_{ij} between O_i and D_j, so that the total cost $C = \sum_{i,j} c_{i,j} x_{ij}$, is minimal, continuing to satisfy the balance conditions. The problem falls within linear programming and can be solved by the simplex method, at the price of greatly increasing the number of unknowns. Why is there a faster method? Even the amateur

guesses a logical thread in the fact that it is easy to construct a feasible solution. However, in general it is not the best solution and therefore needs to be improved. The direct procedure is rather easy to grasp for a manual computation, since a graph can help the intuition, but becomes more complex on the computer that has no eyes. In the paper we sketch also this approach, useful in problems of great dimensions, where the main road of the simplex method cannot be applied in view of the huge dimension that it should attain. The generalizations of the optimal transport methods cannot be afforded in this introductory paper, but some selected references can properly drive the curious student.

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