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APPLICATION OF THE RASCH METHOD OF EVALUATING LATENT VARIABLES IN MANAGEMENT AND ADMINISTRATION

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Abstract. *In the paper approaches of application of the theory of latent variables for the decision of some problems of management and management are offered. The peculiarity of this work is that mathematical solutions for solving problems are based on Rasch's model for estimating latent variables.*
The aim of this paper is to describe a general approach to estimating latent variables using Rasch's method, based on the method of least squares, and apply this approach to some management tasks. The tasks of applying the Rasch model to the method of organizing team, to evaluating alternatives in decision-making and to the formation of a portfolio of securities were solved.
In the field of labour management, three models for organizing group tasks are considered: the formation of work teams, the case of individual performance of a group task, and the case of joint performance of group task jobs. In the field of decision theory, the model for choosing the best alternative is considered, including taking into account the weights of the criteria. We also considered the approach of obtaining estimates of alternatives using the hierarchy analysis method, in which the attractiveness vector of alternatives is computed on the basis of Rasch's model of estimation of latent variables. In the field of financial management, a method of forming a portfolio of securities in the approach of the theory of latent variables is proposed.
It is shown that, in comparison with traditional methods, the approach based on Rasch's model has advantages: linearity of the obtained estimates, their independence and high accuracy.
Keywords: *decision-making, latent variables, the organization of labour, a portfolio of securities, Rasch's model.*

Introduction

Mathematical methods of measuring latent variables is the direction in mathematical modelling that is quite new and not completely investigated. The ability to objectively and adequately measure latent variables will allow to operate at a mathematical level with such qualitative categories as efficiency, level,

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degree, quality, and others. This determines the urgency of using the theory of latent variables in many areas of science, including in management tasks.

The most complete and objective model for estimating latent variables is the Rasch model (Linacre, 2004; Engelhard, 2013) at the present time. Initially, it was used in education to assess the quality of the knowledge received and the indicator variables were to be measured on a dichotomous or political scale. However, at the present time, there have been modifications of the Rasch model is based on the method of least squares, which made it possible to use indicator variables measured by continuous scales. This allowed us to significantly expand the scope of the Rasch model.

The aim of this paper is to describe a general approach to estimating latent variables using Rasch's method, based on the method of least squares (Barkalov et al., 2014; Moiseev, 2015), and apply this approach to some management tasks.

The following problems are considered in the article.

1. Application of the Rasch's model in the problems of labor management. We are considering a set of works (a project) that a group of performers must perform. Three models of organizing group tasks are considered:
   a) the formation of work collectives - when the group of performers is divided into teams with different degrees of responsibility and within each the roles of participants are determined;
   b) the case of individual performance of a group task - when each performer is assigned only one job;
   c) the case of joint performance of group tasks - when several performers can do one job at a time.
2. Application of the model in the theory of decision-making. The approaches to decision-making under conditions of certainty are given. The cases of choosing the best alternative without taking into account the weights of the criteria and taking into account the weights are considered.
3. Application of the model in expert evaluation. An approach is described for obtaining estimates of the attractiveness of alternatives using the hierarchy analysis method, based on the Rasch's model of estimation of latent variables.
4. Application of the model in financial management. The approach of forming a portfolio of securities based on the Rasch's method is proposed.

Here are the main points of Rasch's model.

Theoretical foundations of Rasch's model

The classical Rasch model is based on the logistic function, which makes it possible to find the probability that there will be a positive evaluation by the *i*-th subject of the *j*-th object, if these objects and subjects are evaluated by latent variables β and θ (Maslak, 2005):

$$p_{ij} = \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}}. \quad (1)$$

These estimates are measured in some dimensionless quantities, the scale of which is linear. The estimation of latent variables is made on the basis of known values of indicator (measured) variables x_{ij}.

However, the model is not devoid of shortcomings and limitations. The main drawback of the classical Rasch model is the limited use of the original data. From the point of view of mathematics, this limitation is due to the fact that the computational basis of the model is the maximum likelihood method.

The authors developed a new approach to the calculation of estimates of latent variables, according to which the computational part is based on the least squares method (LSM): the parameters θ_i and β_j for the latent variables estimation model are chosen so that the sum of the squares of deviations of the indicator variables x_{ij} from the estimated probabilities p_{ij} were the smallest.

The task is reduced to minimizing the residual amount (Moiseev, 2015):

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij} - p_{ij})^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_{ij} - \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}}\right)^2 \to \min. \quad (2)$$

In the case of normalizing the estimates to nonnegative values, the objective function (2) is supplemented by a constraint system:

$$\theta_i \geq 0; \ \beta_j \geq 0; \ i = 1,2,...,m; \quad j = 1,2,...,n \quad (3)$$

The main advantage of this model is that it can use a continuous set x_{ij} as the empirical data, having the sense of the degree of evaluation by the *i*-th subject of the *j*-th object whose elements can vary continuously from 0 to 1.

Another important advantage is that the proposed approach significantly expands the instrumental possibilities of solving the problem. The proposed model is a classical problem of non-linear optimization, the numerical solution of which can be realized with the help of available application programs, for example in MS Excel using the Solver add-on.

Application of the model to the problems of labour management

We are considering a set of works (a project) that a group of performers must perform. At the first stage, there is an analysis of the ability of performers to perform a particular job, and then, according to some methodology, the assignment of performers to work takes place. Usually, the project contains several works, each work can be performed by either one or several performers. Let's consider three models for organizing group tasks.

Model for the formation of work teams. Let the number of works in the project m, and the number of performers in the group n. At the first stage, a survey or test is performed, allowing you to preliminarily evaluate whether each performer can perform each task. As a result, we obtain the matrix: $x_{ij} = 1$ if the i-th performer can carry the j-th task, and $x_{ij} = 0$ if the i-th performer can not carry out the j-th tasks.

This matrix is processed by Rasch's method of estimating latent variables. As a result, we obtain estimates of the parameters θ_i – the level of the abilities of the i-th performer, and β_j – the difficulty of the j-th job. On the basis of the estimates obtained, one can find the probability p_{ij} of correct execution of the i-th test by the j-th work, which is determined by the logistic function (1).

These assessments can already be used to identify strong and weak performers and complex and easy works. The probabilities (1) show the assessments of the ability of performers to perform specific jobs, but not the entire task as a whole, except for this there is no estimate of such a final indicator as the probability of the task by the whole group.

Consider the continuation of the methodology for assigning works to performers, eliminating these shortcomings.

At the heart of this model, which makes it possible to divide a group of participants into subgroups with different responsibilities and roles, is a model based on the theory of pair matrix games. According to it, the group of performers plays the role of the first player, the winning of which is maximized. As the payment matrix of the game is a matrix of probabilities p_{ij} (1). One of the common methods for solving game problems is to reduce the model to the problem of linear programming. We introduce some variables x_i, $i=1,2,\dots,n$, for which the objective function and constraints are as follows:

$$\sum_{i=1}^{n} x_i \rightarrow \min;$$
$$\sum_{i=1}^{n} p_{ij} x_i \geq 1,\ j = 1,2,..,m; \quad (4)$$
$$x_i \geq 0,\ i = 1,2,..,n.$$

From the solution (4) we find the optimal values of the variables x_i*, on the basis of which we determine the price of the game: $\nu = 1 \Big/ \sum_{i=1}^{n} x_i^*$ and the probabilities of strategies in the mixed: $P_i = \nu x_i^*$.

Probability data can be interpreted as the optimal shares of performers' participation in a general task or, in other words, as the degree of confidence in the participant, its reliability. The price of the game ν makes sense for the whole group to complete the whole task.

This approach allows us to break up a group of performers into subgroups. From the solution of the game problem, the first, most responsible group, to which the active strategies of the game problem correspond, is singled out. Then, from the remaining participants, the problem of game theory is again created, the solution of which allows us to single out the next active group, but with less responsibility, etc.

The model of individual assignment of performers to work, namely, the individual performance of the work of the group task, each work is performed by only one performer and vice versa, each performer performs only one work.

Consider the case when there are *n* performers who need to perform *n* jobs. At the same time one artist is assigned to one work and one participant can perform only one work. It is necessary to distribute the performers so that the total probability of the entire project is maximized.

From the statement of the problem it is obvious that we have a typical assignment problem. At the first stage, estimates are obtained for Rasch's model, similar to the previous problem. As a matrix of wins with a maximum value, we will use the probability matrix (1), but it must be square. Denote by y_{ij} the assignment matrix, which is defined as:

Then, the mathematical model of the problem will have the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij} y_{ij} \to \max;$$

$$\sum_{i=1}^{n} y_{ij} = 1,\ j = 1,2,..,n; \quad \sum_{j=1}^{n} y_{ij} = 1,\ i = 1,2,..,n; \tag{5}$$

$$y_{ij} \ge 0,\quad y_{ij} - \text{integer},\quad i = 1,2,..,n,\quad j = 1,2,..,n.$$

The solution of this problem will give an optimal distribution of performers for the work of the group task. As an estimate of the likelihood of the entire project, you can use the average of the probabilities of doing all the work.

Model of joint performance of group task jobs. Let's consider now a situation where each participant can perform each of the works in a certain proportion and every work is done by all performers, also in a certain proportion. These shares can be calculated using the approach to solving the transport problem with the maximization of the objective function. The matrix y_{ij} is introduced in the same way as the assignment task, but it will not be discrete and its values will have the meaning of the share of the subject *i* n the work with the number *j*. For the initial data, we take the probability matrix (1).

The mathematical model for solving such a problem will differ from (5) by the absence of the condition of integer variables and the number of jobs is not necessarily equal to the number of performers. If the open transport problem is solved and so are the main constraints of the model (5), the equalities will be changed to the inequalities.

Application of the model in decision theory

Consider the original approach to decision-making under conditions of certainty, based on the theory of estimating latent variables.

Let us first consider the case of making decisions under conditions of certainty without taking into account the weights of the criteria. Let the person making the decision have *n* alternatives $A_1, A_2, \ldots, A_n$ and *m* of the criteria $K_1, K_2, \ldots K_m$. Let U_{ij} us denote the estimate of the *i*-th alternative by the *j*-th criterion. To bring the estimates to a single scale, a normalization procedure is performed, as a result of which all the estimates of the alternatives according to the criteria u_{ij} take values from the interval from 0 to 1. Taking these estimates as the initial data, we find the exponents θ_i and β_j by formulas (2) and (3), which make sense: θ_i - estimates of the attractiveness of alternatives, β_j - the meaning of the degree of satisfiability of the criteria.

We now consider the possibility of taking into account the weights of the criteria. We denote by w_j the weight of the *j*-th criterion. We will assume that the weight is measured on a scale from 0 to 1 and, the more weight, the greater the importance for the decision maker and the greater contribution to the attractiveness of alternatives it should give. There are three approaches to accounting for weights.

Approach 1. We multiply the estimates of the alternatives according to the u_{ij} criteria by their weights w_j, and to calculate the objective function (2) we use $x_{ij} = \tilde{u}_{ij} = u_{ij} w_j$.

Approach 2. When minimizing the residual amount, each term (2) will be taken into account proportionally to the corresponding weight. As a result, an optimization problem of the form:

$$S(\theta_i, \beta_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_j \cdot (u_{ij} - P_{ij})^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} w_j \cdot \left(u_{ij} - \frac{e^{\theta_i - \beta_j}}{1 + e^{\theta_i - \beta_j}} \right)^2 \to \min .$$

Approach 3. As mentioned above, the parameters β_j have the meaning of the degree of satisfiability of the criteria. If we fix these parameters and set them equal to weights, we obtain the problem of finding the attractiveness of the alternatives θ_i taking into account the degree of satisfiability of the criteria: the higher the importance of the criterion, the greater the degree of its feasibility. The mathematical model of the optimization problem will in this case be of the form:

$$S(\theta_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(u_{ij} - \frac{e^{\theta_i - w_j}}{1 + e^{\theta_i - w_j}} \right)^2 \to \min.$$

The methodology used in this paper allows one to obtain estimates that, on the one hand, are in good agreement with classical assessments, but, on the other hand, they are more flexible, making it possible to make an unambiguous decision in situations where traditional approaches do not give it, and are more resistant to small changes baseline data.

Application of the model in expert evaluation

In the theory of decision-making, analytical planning and expert evaluation, one of the most important tasks is the quantitative assessment of qualitative indicators.

To solve a similar problem, the hierarchy analysis method (Analytic Hierarchy Process – AHP), developed by T. Saaty (1988), and its modification the multiplicative AHP (Lootsma & Schuijt, 1997) are employed. These methods are based on a pairwise comparison of alternatives using a verbal scale of relative importance. The results of the comparison are translated into some quantitative indicators of the attractiveness of alternatives in accordance with a given scale.

The authors propose an original model for estimating multicriteria alternatives, based on the described model for estimating latent variables.

Let there be *n* alternatives that need to be estimated on a quantitative scale by some qualitative criterion. Suppose that the expert tries to determine this estimate by means of paired comparisons. In the classical approach, they are the degrees of superiority of alternatives, and the vector of preferences for this criterion.

Consider now an approach based on a model for estimating latent variables. According to it, the probability P_{ij} of choosing the alternative A_i in comparison with the alternative A_j can be defined as:

$$P_{ij} = \frac{e^{\beta_i - \beta_j}}{1 + e^{\beta_i - \beta_j}}. \tag{6}$$

To apply (8) in practice, it is necessary to find the estimates β_i based on the known probabilities P_{ij} of alternative preferences, which are obtained empirically by means of expert comparison of the preferences. Let p'_{ij} be the probability that the expert will choose an alternative to A_i versus alternative A_j. Taking these probabilities as input data for the model of estimation of latent variables, we obtain the optimization problem:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(p'_{ij} - P_{ij})^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(p'_{ij} - \frac{e^{\beta_i - \beta_j}}{1 + e^{\beta_i - \beta_j}}\right)^2 \to \min. \tag{7}$$

If we use a positive scale of estimates, then the normalization condition can be the equality of the smallest estimate β_i zero:

$$\min_i \beta_i = 0. \tag{8}$$

After the estimates of the attractiveness of the alternatives are found, we can calculate their weights w_i for this criterion by normalizing β_i. If condition (8) is used, then the normalization to the unit scale can be carried out according to formulas: $w_i = \frac{\beta_i}{\max_i \beta_i}$ or $w_i = \frac{\beta_i}{\sum_{i=1}^{n} \beta_i}$.

Let us return to the question of finding empirical probabilities p'_{ij}. According to the AHP method, there is a scale of preferences, which for the probabilistic approach can be specified in accordance with Table. 1. If there is a superiority of the second alternative over the first, then its probability will be reverse: p'_{ij}=1- p'_{ij}, p'_{ii}=0,5.

Table 1. **Scale of relative importance** (Barkalov et al., 2014)

Level of importance of the 1st alternative over the 2nd	Probability p'_{ij}
Equal importance	0,5
Moderate superiority	0,6
Substantial superiority	0,7
A significant, great superiority	0,8
Very great superiority	0,9
Unambiguous preference	1

The described method has advantages over traditional ones: estimates of alternatives are their unique properties, not depending on the alternative, and are measured on a linear scale.

Application of the model in financial management

The approach described above can also be used in financial management, in particular, in the formation of a securities portfolio through an individual expert evaluation of the appropriateness of including a particular security in a portfolio.

At the first stage, all securities that can be included in the portfolio are selected. Let the number of such papers n be denoted by A_1, A_2, …, A_n. To assess the effectiveness of the inclusion of securities in the portfolio, an expert evaluation of each of them is made. Suppose that m experts B_1, B_2, …, B_m participate in the work. Each expert makes an assessment of the appropriateness of including a security in the portfolio, forming a matrix: x_{ij} - the degree of preference for including an A_i security in the portfolio in the opinion of the j-th expert. The valuation is measured on a scale of 0 to 1 and it can be interpreted as the probability or share with which the expert would include the security in the portfolio.

Next, we introduce latent variables: θ_i - the degree of attractiveness of the i-th security from the point of view of its inclusion in the portfolio; β_j - some indicator that characterizes the rigor (loyalty) of the j-th expert. To obtain estimates, we solve the optimization problem (2) and (3).

At the final stage, based on the obtained estimates of the latent variables θ_i, the portfolio is formed. The shares (weights) q_i of securities in the block are obtained by normalizing the estimates θ_i:

$$q_i = \frac{\theta_i - \theta_{\min}}{\sum_{i=1}^{n} (\theta_i - \theta_{\min})} \ . \tag{9}$$

Estimates of the composition of the securities portfolio correlate well with known methods, such as models for the formation of a block of securities, such as, for example, the Markovitz model or the price-level model (Burenin, 2000), however, estimates based on Rasch's model have advantages: assessments of the attractiveness of securities are their unique properties and do not depend on a set of experts and estimates on a linear scale: in addition to valuations of securities, it is possible to obtain assessments of the quality of the work of experts β_j.

Conclusion

A new approach to the estimation of latent variables by Rasch's model is proposed. It is based on the application of the method of least squares in the computational kernel instead of the maximum likelihood method. This will allow us to use as a source data continuous sets of values of indicator variables.

Computational experiments have shown that the model based on LSM is characterized by high statistical accuracy and it can be used to calculate estimates of the latent variable.

The properties of estimates of latent variables obtained by the method of least squares are analysed in research by Maslak, Moiseev & Osipov (2015) and Maslak, Moiseev, Osipov & Pozdnyakov (2017) supports. Estimates of the parameters of the model based on the least squares method have a smaller error in calculating the estimates than those calculated using the maximum likelihood method. At the same time, the proposed model has a much wider range of applications than the classical Rasch model. Let's consider some directions of application of the model in various areas of management and management.

At the same time, this model has a much wider range of practical applications than the classical model of Rasch.

On the basis of Rasch's model based on LSM, new approaches to solving the problems of labour organization, decision-making, expert appraisal, and the formation of a security portfolio were proposed.

In conclusion, we draw attention to the fact that the approach to using the Rasch model based on LSM in management tasks can be applied to other scientific areas.

References

Barkalov, S. A., Kireev, Yu. V., Kobelev, V. S., & Moiseev, S. I. (2014). Model assessment of the attractiveness of alternatives in the approach of Rasch model. *Sistemy upravleniya i informatsionnyye tekhnologii* [Control Systems and Information Technology], *3.2* (57), 209-213 (in Russian).

Barkalov, S. A., Moiseev, S. I., & Solovieva, E. V. (2014). Application of the Least Squares Method of When Estimating Latent Variables by the Rasch Method. *Scientific herald of the Voronezh State University of Architecture and Civil Engineering. Ser. "Construction management"*, *1* (6), 98-100 (in Russian).

Burenin, A. N. (2000). Market of Securities and Production Financial Instruments. *Moscow: The Federal Book Trade Company* (in Russian).

Engelhard, G. Jr. (2013). Invariant measurement using Rasch models in the social, behavioral, and health sciences / G. Engelhard Jr. – *N.Y.: Routledge*.

Linacre, J. M. (2004). Estimation methods for Rasch measures. *In Introduction to RasczRh measuremen*, ed. Smith E.V., Jr, Smith R.M, 25-47.

Lootsma, F. A., & Schuijt, H. (1997). The Multiplicative AHP, SMART and ELECTRE in a Common Contex. *J. Multi-Criteria Decision Analysis*, *6*.

Maslak, A. A. (2005). Measuring and comparing higher education quality between countries worldwide / A. A. Maslak [et al.] *Journal of Applied Measurement*, *6*, 4, 432-442.

Maslak, A. A., Moiseev, S. I., & Osipov, S. A. (2015). Comparative Analysis of the Estimates of Rasch Model Parameters, Obtained by Maximum Likelihood and Least Squares. *Management Problems*, *5*, 58-66 (in Russian).

Maslak, A. A., Moiseev, S. I., Osipov, S. A., & Pozdnyakov, S. A. (2017). A Research of Measurement Precision of Latent Variable Depending on the Range of Variation of an Indicators Set. *Radio Electronics, Computer Science, Control*, *1*, 40-48.

Moiseev, S. I. (2015). Rasch Model Estimates of Latent Variables Based on the Method of Least Squares. *Economics and Management Control Systems, 2.1* (16), 166–172 (in Russian).

Saaty, T. L. (1988). Decision Making for Leaders: The Analytical Hierarchy Process for Decisions in a Complex World, *Wadsworth. Paperback, RWS.*