SIMPLE METHODS OF ENGINEERING CALCULATIONS FOR SOLVING TRANSFER PROBLEMS OF MULTI – SUBSTANCES IN HORIZONTAL LAYER

VIENKĀRŠU ALGORITMU IZSTRĀDE DAUDZKOMPONENTU MATERIĀLU PĀRNESES PROBLĒMAI HORIZONTĀLĀ SLĀNĪ

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Abstract. *We consider the simple algorithms in the modelling of the transfer problem of different substances (concentration, heat, moisture, and e. c.) in plate. The approximations of corresponding initial – boundary value problem of the system of the partial differential equations (PDE) is based on the finite volume method. This procedure allows one to reduce the 2-D transfer problem described by a PDE to initial value problem for a system of ordinary differential equations (ODE) of the first or second order. In the stationary case the exact finite – difference vector scheme is obtained.*

Keywords: *2-D heat transfer problem, finite – volume method, finite – difference scheme, diffusion of heat and moisture.*

Introduction

By means of the differential equations of parabolic type (for example, the equations of heat conductivity $\partial(K\partial u/\partial x)/\partial x + F(x,t) = c\rho\partial u/\partial x)$ and equations of Poison type $u_{xx} + u_{yy} = f(x,y), \quad (x,y)\in D$ considering them with boundary and initial conditions become possible the mathematical modelling of many theoretical and practical problems. Their solutions serve not only as theoretical base for the further researches, but also are decisions of many actual practical problems.

For example, it is possible to estimate the heat-shielding properties of protections of the equipment at influence of a fire, to calculate of dynamics of dangerous factors of a fire in a room [9]. The calculation of heat transfer in a ground layer and in an atmosphere [10] is possible. It is possible to find out the algorithm for describing the temperature changes depending on time for surfaces on fibbers glasses at its heating without radiation [3] and with radiation [4,5], the algorithm for calculation of a temperature mode in a wall of a building with linings taking into account an opportunity of there heating [6,7].

Here it is very important to find simple algorithms for engineering - technical calculations, which are characterized by the sufficient accuracy of calculations, by the simplicity of calculations (with an opportunity of use of mathematical systems of a high level, for example, Mathematica, Maple, Matlab, etc.), by the universality of algorithm (the decision not only the given problem, but also an opportunity of its application for the decision of wider class of adjacent questions).

One of ways of the decision of the named problem is the reducing of a differential problem to system of the ordinary differential equations using thus, for example, a method of final volumes [2].

Continuing beforehand mentioned subjects we shall consider the forming of simple algorithms for modelling of the transfer problem of different substances (concentration, heat, moisture, and e. c.) in plate.

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Materials and methods

Mathematical model

Let's consider transfer problem of m substances $(m \geq 2)$ in the plate $\Omega = \{0 \leq x \leq l, -\infty < y < +\infty, -\infty < z < +\infty\}$ with thickness l. We shall consider the initial – boundary value problem of system m PDE -s for vector – function (vector – column) $u = u(x,t) = \{u^{(1)}(x,t), u^{(2)}(x,t), ..., u^{(m)}(x,t)\}^T$ in the following form

$$G\frac{\partial u}{\partial t} = L\frac{\partial^2 u}{\partial x^2} - Q(x,t), \qquad (1.1)$$

where G is quadratic – matrix $m \times m$ with constant elements $g^{(i,j)}$ $(\det(G) \neq 0)$, L is quadratic, positive definite matrix $m \times m$ with constant elements $l^{(i,j)} > 0$, Q is vector – column $m \times 1$ with elements $q^{(i,j)}(x,t), i, j = \overline{1,m}$. The boundary conditions on the surfaces $x = 0, x = l$ are

$$L\frac{\partial u(0,t)}{\partial x} = \alpha_0(u(0,t) - T_0),\ L\frac{\partial u(l,t)}{\partial x} = \alpha_l(T_l - u(l,t)), \qquad (1.2)$$

where α_0, α_l are the positive definite matrix (transfer matrix) with constant elements $\alpha_0^{(i,j)}, \alpha_l^{(i,j)}$, T_0, T_l are known vector – functions with constant elements $T_0^{(j)}, T_l^{(j)}$, $j = \overline{1,m}$.

For the initial conditions for $t = 0$ we give

$$u(x,0) = \varphi(x), \qquad (1.3)$$

where $\varphi(x)$ is vector – column with elements $\varphi^{(j)}(x), j = \overrightarrow{1,m}$.

If the elements of matrix α_0, α_l are equal infinity, then we have the first kind boundary conditions in the form

$$u(0,t) = T_0,\ u(l,t) = T_l \qquad (1.4)$$

The vector transfer equation (1.1) can be presented in the following form

$$L\frac{\partial^2 u}{\partial x^2} = F, \qquad (1.5)$$

where the vector column $F = G\dot{u} + Q$, $\dot{u} \equiv \frac{\partial u}{\partial t}$. If $\dot{u} = 0$ and $Q = Q(x)$, then we have the stationary vector boundary – value problem (1.2), (1.5).

The exact vector 3 – points finite difference scheme in the stationary case

We use the method of finite volumes [2-2] for approximation of the differential problem (1.1) – (1.4). We consider $N+1$ grid points in the x - direction $0 < x_0 < x_1 < ... < x_N = l$ with steps $h_k = x_k - x_{k-1}, k = \overline{1,N}$. The exact vector finite – difference scheme for given vector – function F can be obtained in similar [2, 5] form

$$\begin{cases} Lh_1^{-1}(u_1 - u_0) - \alpha_0(u_0 - T_0) = \tilde{R}_0^+ \\ Lh_{k+1}^{-1}(u_{k+1} - u_k) - Lh_k^{-1}(u_k - u_{k-1}) = \tilde{R}_k,\ k = \overline{1, N-1}, \\ \alpha_l(T_l - u_N) - Lh_N^{-1}(u_N - u_{N-1}) = \tilde{R}_N^- \end{cases} \qquad (2.1)$$

where $u_k = u(x_k,t) \equiv u_k(t)$, $\tilde{R}_k = \tilde{R}_k^+ + \tilde{R}_k^-, k = \overline{1,N-1}$, $\tilde{R}_k^+ = GR_k^+ + I_k^+$, $\tilde{R}_k^- = GR_k^- + I_k^-$,

$$R_k^- = \frac{1}{h_k}\int_{x_{k-1}}^{x_k}(x - x_{k-1})\dot{u}_k(x,t)dx, \qquad R_k^+ = \frac{1}{h_{k+1}}\int_{x_k}^{x_{k+1}}(x_{k+1} - x)\dot{u}_{k+1}(x,t)dx,$$

$$I_k^- = \frac{1}{h_k}\int_{x_{k-1}}^{x_k}(x - x_{k-1})Q(x,t)dx, \quad I_k^+ = \frac{1}{h_{k+1}}\int_{x_k}^{x_{k+1}}(x_{k+1} - x)Q(x,t)dx$$

are the vectors – column of m order.

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If Q is the constant vector, then integrals are in form $I_k^- = 0.5h_k Q$, $I_k^+ = 0.5h_{k+1}Q$.

In the non-stationary case $(\dot{u} \neq 0, Q = Q(x,t))$ we must do integrals $R_k^\pm$ approximately with corresponding quadrate formulas. Now we shall discuss only 3 grid points $x_0 = 0$ $x_1 = h_1$, $x_2 = l = h_1 + h_2$ $(N = 2)$.

The 3 - grid points problem and approximation of integrals

The vector finite – difference scheme ($\alpha_0 = \propto$ - the elements of matrix α_0 are equal to infinity or in the first boundary condition (1.2) $u(0,t) = T_0$) is in the form

$$\begin{cases} Lh_2^{-1}(u_2 - u_1) - Lh_1^{-1}(u_1 - T_0) = G(R_1^+ + R_1^-) + I_1 \\ \alpha_l(T_l - u_2) - Lh_2^{-1}(u_2 - u_1) = GR_2^- + I_2^- \end{cases} \tag{3.1}$$

where $I_1 = I_1^- + I_1^+$, $R_1^+ = \dfrac{1}{h_2}\int\limits_{h_1}^{l}(l-x)\dot{u}(x,t)dx = h_2 J_1$, $R_2^- = \dfrac{1}{h_2}\int\limits_{h_1}^{l}(x-h_1)\dot{u}(x,t)dx = h_2 J_2$,

$$J_1 = \int\limits_0^1 (1-\tilde{x})V_2(\tilde{x})d\tilde{x} \ , \qquad J_2 = \int\limits_0^1 \tilde{x}V_2(\tilde{x})d\tilde{x} \ , \qquad \tilde{x} = (x-h_1)/h_2, \qquad V_2(\tilde{x}) = \dot{u}(h_1 + \tilde{x}h_2, t),$$

$$R_1^- = \frac{1}{h_1}\int\limits_0^{h_1} x\dot{u}(x,t)dx = h_1 J_3 \ , \ J_3 = \int\limits_0^1 \tilde{x}V_1(\tilde{x})d\tilde{x} \ , \ \tilde{x} = x/h_1, \ V_1(\tilde{x}) = \dot{u}(\tilde{x}h_1, t).$$

In the non-stationary case $(\dot{u}_k \neq 0)$, using initial – value problem for system of ODE one must do integrals R_1^+, R_1^-, R_2^- approximately with quadrature formulas contained the derivatives of the first and second order in following way:

$$J_k = A_1^{(k)}V_2(0) + A_2^{(k)}V_2(1) + A_3^{(k)}V_2'(1) + B_1^{(k)}V_2''(0) + B_2^k V_2''(1) + r_k, \ k = 1,2 \tag{3.2}$$

$$J_3 = A_1^{(3)}V_1(0) + A_2^{(3)}V_1(1) + B_1^{(3)}V_1''(0) + B_2^{(3)}V_1''(1) + r_3 \tag{3.3}$$

where $r_k = \dfrac{h_2^5}{5!}\partial^5\dot{u}(\xi_k,t)/\partial x^5 C_k$, $\xi_k \in [h_1, l]$, $k = 1,2$, $r_3 = \dfrac{h_1^4}{4!}\partial^4\dot{u}(\xi_k,t)/\partial x^4 C_3$, $\xi_3 \in [0, h_1]$,

are the vectors – error terms, $A_j^{(k)}, B_j^{(k)}, C_k$ $(j,k = 1,2,3)$ are the indefinite coefficients.

The coordinates of vectors $J_k, k = 1,2,3$, $V_1(0)$, $V_2(0)$, $V_1(1)$, $V_2(1)$, $V_2'(1)$, $V_1''(0)$, $V_1''(1)$, $V_2''(0)$, $V_2''(1)$ are independent of the coefficients of quadrature formulas and we can determine the coefficients using the scalar power functions $V_1(\tilde{x}) = \tilde{x}^i$, $i = \overline{0,4}$, $V_2(\tilde{x}) = \tilde{x}^i$, $i = \overline{0,4}$. We get the system of linear algebraic equations for $A_j^{(k)}$, $B_j^{(k)}$ in the form

$$1/(i+1)(i+2) = A_1^{(1)}0^i + A_2^{(1)} + iA_3^{(1)} + i(i-1)\left(B_1^{(1)}0^{i-2} + B_2^{(1)}\right); \tag{3.4}$$

$$1/i+2 = A_1^{(2)}0^i + A_2^{(2)} + iA_3^{(2)} + i(i-1)\left(B_1^{(2)}0^{i-2} + B_2^{(2)}\right); \tag{3.5}$$

$$1/(i+2) = A_1^{(3)}0^i + A_2^{(3)} + i(i-1)\left(B_1^{(3)}0^{i-2} + B_2^{(3)}\right); \tag{3.6}$$

where $0^j = 1, \ j \leq 0$.

The solutions of these systems are

$A_1^{(1)} = 7/30$, $A_2^{(1)} = 4/15$, $A_3^{(1)} = -(1/10)$ $B_1^{(1)} = -(1/180)$, $B_2^{(1)} = -(1/72)$, $A_1^{(2)} = 1/15$, $A_2^{(2)} = 13/30$, $A_3^{(2)} = -(1/10)$, $B_1^{(2)} = -(1/360)$ $B_2^{(2)} = 1/90$, $A_1^{(3)} = 1/6$, $A_2^{(3)} = 1/3$, $B_1^{(3)} = -(7/360)$, $B_2^{(3)} = -(1/45)$.

Constants C_k in the residual r_k $(k = \overline{1,3})$ are determined from (3.2), (3.3) if $V_1(\tilde{x}) = \tilde{x}^4$, $V_2(\tilde{x}) = \tilde{x}^5$:

$C_1 = -(13/630)$, $C_2 = -(4/315)$, $C_3 = 1/10$.

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Results and discussion

By means of the considered algorithm it is possible to receive the following results:

1. In the case of $\alpha_0 \neq \infty$ we can in the formula (3.3) add the term $A_3^{(3)}V_1'(0)$ and in the formula (3.6) the term $iA_3^{(3)}0^{i-1}$; then $A_1^{(3)} = 4/15$, $A_2^{(3)} = 7/30$, $A_3^{(3)} = 1/10$, $B_1^{(3)} = 1/72$, $B_2^{(3)} = -(1/180)$, $C_3 = 13/630$, $r_3 = O(h_1^5)$;

2. If the formula (3.2), (3.3) contains the derivative of the first order $\left(B_1^{(k)} = B_2^{(k)} = 0\right)$, then $A_1^{(1)} = A_2^{(1)} = 1/4$, $A_3^{(1)} = -(1/12)$, $C_1 = 1/20$; $r_1 = O(h_2^3)$, $A_1^{(2)} = 1/12$, $A_2^{(2)} = 5/12$, $A_3^{(2)} = -(1/12)$, $r_2 = O(h_2^3)$, $A_1^{(3)} = 1/6$, $A_2^{(3)} = 1/3$, $C_3 = -(1/12)$, $r_3 = O(h_1^2)$.

3. If the integrals J_1 are approximated without the derivatives $\left(A_3^{(1)} = B_1^{(1)} = B_2^{(1)} = 0\right)$, then from (3.4) we obtain [7, 8]
$A_1^{(1)} = 1/3$, $A_2^{(1)} = 1/6$, $C_1 = -(1/12)$; $r_1 = O(h_2^2)$

4. If the integrals J_3 (similar J_2) are approximated without the derivatives of second order and $\alpha_0 \neq \infty$ (in the formula (3.3) we have term $A_3^{(3)}V_1'(0)$) then from (3.6) we obtain
$A_1^{(3)} = A_2^{(3)} = 1/4$, $A_3^{(3)} = 1/12$, $C_3 = -(1/20)$; $r_3 = O(h_1^3)$;

5. In the case of $\alpha_0 \neq \infty$ we have by (3.1) adding from (2.1) vector difference equations
$Lh_1^{-1}(u_1 - u_0) - \alpha_0(u_0 - T_0) = GR_0^+ + I_0^+$,

where $R_0^+ = \dfrac{1}{h_1}\displaystyle\int_0^{h_1}(h_1 - x)\dot{u}(x,t)dx = h_1 J_0$, $J_0 = \displaystyle\int_0^1 (1-\tilde{x})V_1(\tilde{x})d\tilde{x}$ - and in the first equations (3.1) replace vector T_0 by u_0. Then

$J_0 = A_1^{(0)}V_1(0) + A_2^{(0)}V_1(1) + A_3^{(0)}V_1'(0) + B_1^{(0)}V_1''(0) + B_2^{(0)}V_1''(1) + r_0$,

where $r_0 = \dfrac{h_1^5}{5!}\partial^5\dot{u}(\xi_0,t)/\partial x^5 C_0$, $\xi_0 \in [0, h_1]$

In this case we have the equations for $A^{(0)}, B^{(0)}$ in the form

$1/(i+1)(i+2) = A_1^{(0)}0^i + A_2^{(0)} + iA_3^{(0)}0^{i-1} + i(i-1)\left(B_1^{(0)}0^{i-2} + B_2^{(0)}\right)$, $i = \overline{0,4}$,

where $A_1^{(0)} = 13/30$, $A_2^{(0)} = 1/15$, $A_3^{(0)} = 1/10$ $B_1^{(0)} = 1/90$, $B_2^{(0)} = -(1/360)$.

If $V_1(\tilde{x}) = \tilde{x}^5$, then $C_0 = 4/315$.

Using the vector difference equation (3.1) and the right – side integral approximation (3.2), (3.3) with the neglected error terms $r_k, k = 1,2,3$, the approximate numerical solutions – vectors $u_1(t), u_2(t)$ at every time moment $t > 0$ can be found by solving the following vector system of ODE – s of second order $(\dot{u}_0 = \ddot{u}_0 = 0, \alpha_0 = \infty)$:

$$\begin{aligned}&Gh_2\left(A_1^{(1)}\dot{u}_1 + A_2^{(1)}\dot{u}_2 - h_2 A_3^{(1)}L^{-1}\alpha_l\dot{u}_2 + h_2^2 B_1^{(1)}L^{-1}\left(\dot{Q}(h_1,t) + 6\ddot{u}_1\right) + h_2^2 B_2^{(1)}L^{-1}\left(\dot{Q}(l,t) + 6\ddot{u}_2\right)\right) + \\ &6h_1\left(A_2^{(3)}\dot{u}_1 + h_1^2 B_1^{(3)}L^{-1}\dot{Q}(0,t) + h_1^2 B_2^{(3)}L^{-1}\left(\dot{Q}(h_1,t) + 6\ddot{u}_1\right)\right) + I_1 = L\left(h_2^{-1}(u_2 - u_1) - h_1^{-1}(u_1 - T_0)\right)\end{aligned} \quad (3.7)$$

$$\begin{aligned}&Gh_2\left(A_1^{(2)}\dot{u}_1 + A_2^{(2)}\dot{u}_2 - h_2 A_3^{(2)}L^{-1}\alpha_l\dot{u}_2 + h_2^2 B_1^{(2)}L^{-1}\left(\dot{Q}(h_1,t) + 6\ddot{u}_1\right) + h_2^2 B_2^{(2)}L^{-1}\left(\dot{Q}(l,t) + 6\ddot{u}_2\right)\right) + \\ &+ I_2^- = \alpha_l(T_l - u_2) - Lh_2^{-1}(u_2 - u_1)\end{aligned} \quad (3.8)$$

The initial conditions are

$$u_1(0) = \varphi(h_1),\ u_2(0) = \varphi(l),\ \dot{u}_1(0) = G^{-1}\left(L\varphi''(h_1) - Q(h_1,0)\right),$$
$$\dot{u}_2(0) = G^{-1}\left(L\varphi''(l) - Q(l,0)\right) \quad (3.9)$$

Here one should take in account that from (1.1 - 1.4) follows: $V_1(0) = \dot{u}_0$, $V_1(1) = \dot{u}_1$, $V_2(0) = \dot{u}_1$, $V_2(1) = \dot{u}_2$, $V_1'(0) = h_1\partial\dot{u}(0,t)/\partial x = h_1\partial u'(0,t)/\partial t = h_1 L^{-1}\alpha_0\dot{u}_0$,
$V_2'(1) = h_2\partial u'(l,t)/\partial t = -h_2 L^{-1}\alpha_l\dot{u}_2$, $V_1''(0) = h_1^2\partial^2\dot{u}(0,t)/\partial x^2 = h_1^2 L^{-1}\left(G\ddot{u}_0 + \dot{Q}(0,t)\right)$

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$V_1''(1) = h_1^2 L^{-1}\left(G\ddot{u}_1 + \dot{Q}(h_1,t)\right)$, $V_2''(0) = h_2^2 \partial^2 \dot{u}(h_1,t)/\partial x^2 = h_2^2 L^{-1}\left(G\ddot{u}_1 + \dot{Q}(h_1,t)\right)$,
$V_2''(1) = h_2^2 L^{-1}\left(G\ddot{u}_2 + \dot{Q}(h_1,t)\right)$, where $u' = \partial u/\partial x$.

The uniform grid in the 3 – points problem

If $h_1 = h_2 = h$, then we can approximate both integrals $R_1 = R_1^+ + R_1^-$ and R_2^- in equations (3.1), in the form

$R_1/h = J_1 = \int_0^1 \tilde{x} V(\tilde{x}) d\tilde{x} + \int_1^2 (2-\tilde{x}) V(\tilde{x}) d\tilde{x}$, $\quad R_2^-/h = J_2 = \int_1^2 (\tilde{x}-1) V(\tilde{x}) d\tilde{x}$, $\quad \tilde{x} = x/h$, where $V(\tilde{x}) = \dot{u}(\tilde{x}h, t)$,

$$J_k = A_1^{(k)}V(0) + A_2^{(k)}V(1) + A_3^{(k)}V(2) + A_4^{(k)}V'(2) + B_1^{(k)}V''(0) + B_2^{(k)}V''(1) + B_3^{(k)}V''(2) + r_k, \quad (4.1)$$

$r_k = (h^7/7!)\partial^7 \dot{u}(\xi_k,t)/\partial x^7 C_k$, $\xi_k \in [0,l]$, $k = 1,2$.

Using the power functions $V(\tilde{x}) = \tilde{x}^i$, $i = \overline{0,6}$ in the expression (4.1) we obtain the solutions of two systems of 7 linear algebraic equations [8] in the form

$A_1^{(1)} = A_3^{(1)} = 11/252$, $A_2^{(1)} = 115/126$, $A_4^{(1)} = 0$, $B_1^{(1)} = B_3^{(1)} = -(13/15120)$
$B_2^{(1)} = 313/7560$, $A_1^{(2)} = 5/204$, $A_2^{(2)} = 13/252$, $A_3^{(2)} = 221/504$, $A_4^{(2)} = -(2/21)$,
$B_1^{(2)} = -(19/30240)$, $B_2^{(2)} = -(37/30240)$, $B_3^{(2)} = 269/30240$, $C_1 = 0$, $C_2 = -(16/315)$.

Therefore $r_1 = (h^8/8!)\partial^8 \dot{u}(\xi_1,t)/\partial x^8 C_1$ and $C_1 = 59/1890$.

Here from $V(0) = \dot{u}_0$, $V(1) = \dot{u}_1$, $V(1) = \dot{u}_2$, $V'(2) = -hL^{-1} - \alpha_l \dot{u}_2$, $V''(0) = h^2 L^{-1}\left(G\ddot{u}_0 + \dot{Q}_0\right)$, $V''(1) = h^2 L^{-1}\left(G\ddot{u}_1 + \dot{Q}_1\right)$, $V''(2) = h^2 L^{-1}\left(G\ddot{u}_2 + \dot{Q}_2\right)$ follows the system of ODE – s in the form

$$Gh^2\left(A_1^{(1)}\dot{u}_0 + A_2^{(1)}\dot{u}_1 + A_3^{(1)}\dot{u}_2 + h^2 L^{-1}\left(B_1^{(1)}\left(G\ddot{u}_0 + \dot{Q}_0\right) + B_2^{(1)}\left(G\ddot{u}_1 + \dot{Q}_1\right) + B_3^{(1)}\left(G\ddot{u}_2 + \dot{Q}_2\right)\right)\right) + hI_1 = L\left(u_2 - 2u_1 + T_0\right) \quad (4.2)$$

$$Gh^2\left(A_1^{(2)}\dot{u}_0 + A_2^{(2)}\dot{u}_1 + A_3^{(2)}\dot{u}_3 - hA_4^{(2)}L^{-1}\alpha_l \dot{u}_2 + h^2 L^{-1}\begin{pmatrix} B_1^{(2)}\left(G\ddot{u}_0 + \dot{Q}_0\right) + B_2^{(2)}\left(G\ddot{u}_1 + \dot{Q}_1\right) + \\ B_3^{(2)}\left(G\ddot{u}_2 + \dot{Q}_2\right)\end{pmatrix}\right) + hI_2^{-1} = h\alpha_l\left(T_l - u_2\right) - L\left(u_2 - u_1\right) \quad (4.3)$$

The following initial conditions are in the form (3.9).

If $\alpha_0 = \alpha_l = \infty$, $u_2 = T_l$, $Q = const$, then $\dot{u}_0 = \dot{u}_2 = 0$, then $\ddot{u}_0 = \ddot{u}_2 = 0$, $\dot{Q} = 0$ and from (4.2) follows the ODE of second order

$$Gh^2 A_2^{(1)}\dot{u}_1 + h^4 B_2^{(1)} G L^{-1} G\ddot{u}_1 + h^2 Q = L\left(T_l - 2u_1 + T_0\right) \quad (4.4)$$

If in the formula (4.1) are only the values $V(0), V(1)$ used, then we have the system of ODE –s of first order

$$h^2\left((2/3)G\dot{u}_1 + Q\right) = L\left(T_l - 2u_1 + T_0\right). \quad (4.5)$$

If in the formula (4.1) are the values $V(0), V(1), V(2)$ used, then we have the following system of ODE –s of first order with error term $O(h^4)$ [7, 8]

$$h^2\left((5/6)G\dot{u}_1 + Q\right) = L\left(T_l - 2u_1 + T_0\right). \quad (4.6)$$

Application example1

Let us consider the case $Q = 0$, $T_0 = T_l = 0$, $\varphi(x) = \left(\sin(\pi x), \sin(\pi x)\right)^T$, $l = 1$, $h = 0.5$, $L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then the exact solution of PDE problem (1.1), (1.3), (1.4) is

$u(x,t) = \left(e^{-\pi^2 t}\sin(\pi x), e^{-\pi^2 t}\left(1 + \pi^2 t\right)\sin(\pi x)\right)^T$ or $u_1 = u(h,t) = \left(e^{-\pi^2 t}, e^{-\pi^2 t}\left(1 + \pi^2 t\right)\right)^T$. (4.7)

From the first order ODE – s (error $O(h^2)$) follows the initial – value problem $G\dot{u}_1 = -12u_1$, $u_1(0) = (1,1)^T$ with the solutions $u_1 = \left(e^{-12t}, e^{-12t}\left(1 + 9.6t\right)\right)^T$. (4.8)

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Therefore in the (4.7) the value π^2 is replaced with 12.
From the ODE – s (4.6) (error $O(h^4)$) follows the equations $G\dot{u}_1 = -9.6u_1$ and the solution is

$$u_1 = \left(e^{-9.6t}, e^{-9.6t}(1+9.6t)\right)^T, \tag{4.9}$$

therefore in the (4.7) the value π^2 is replaced with 9.6.
From the second order ODE – s (4.4) (error $O(h^8)$) follows the initial – value problem

$$\begin{cases} b_1 G^2 \ddot{u}_1 + a_1 G\dot{u}_1 + u_1 = 0 \\ u_1(0) = (1,1)^T,\ \dot{u}_1(0) = -G^{-1}\left(\pi^2,\pi^2\right)^T = \left(-\pi^2, 0\right)^T \end{cases}, \tag{4.10}$$

where $G^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $G^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $a_1 = h^2 A_2^{(1)}/2$, $b_1 = h^4 B_2^{(1)}/2$, $A_2^{(1)} = 115/8.126$,
$B_2^{(1)} = 313/32.7560$, $b_1 = 1/32$, $a_1 = 1/8$.
Let denote $u_1^{(1)} \equiv y$, $u_1^{(2)} \equiv z$, then we have the initial – value system of two ODE – s of second order

$$\begin{cases} b_1\ddot{y} + a_1\dot{y} + y = 0,\ y(0) = 1,\ \dot{y}(0) = -\pi^2 \\ b_1\ddot{z} + a_1\dot{z} + z = -2b_1\ddot{y} - a_1\dot{y},\ z(0) = 1,\ \dot{z}(0) = 0 \end{cases}.$$

The solution is

$$\begin{cases} y(t) = D_1 \exp(\mu_1 t) + D_2 \exp(\mu_2 t) \\ z(t) = D_1(1-\mu_1 t)\exp(\mu_1 t) + D_2(1-\mu_2 t)\exp(\mu_1 t) \end{cases}, \tag{4.11}$$

where $\mu_{1,2} = -a_1/(2b_1) \pm \sqrt{(a_1/2b_1)^2 - 1/b_1}$, $\mu_1 = -9.87$, $\mu_1 = -78.3$,
$D_1 = \left(\mu_2 + \pi^2\right)/\left(\mu_2 - \mu_1\right)$, $D_2 = -\left(\pi^2 + \mu_1\right)/\left(\mu_2 - \mu_1\right)$.
Using the approximation $u''(h,t) \approx \Lambda u_1 = -h^{-2} 2u_1(t)$ (the method of lines with error $O(h^2)$) we get first order ODE – s $G\dot{u}_1 = -8u_1$ and the solution is

$$u_1 = \left(e^{-8t}, e^{-8t}(1+8t)\right)^T, \tag{4.12}$$

therefore in the exact solution (4.7) the value π^2 is replaced with 8.
Using the approximation $u''(h,t) \approx \Lambda u_1 - \left(h^2/12\right)\partial^4 u(h,t)/\partial x^4 = \Lambda u_1 - \left(h^2/12\right)\left(L^{-1}G\right)^2 \ddot{u}_1$ (the method of lines with error $O(h^4)$), we have the problem (4.10) with $b_1 = \left(h^4/24\right) = 1/384$, $a_1 = \left(h^2/2\right) = 1/8$, and the solution is in the form (4.11).
The results of calculation obtained by MAPLE are seen in the Table 1 and Table 2, where u_*, v_* - exact values of $u_1^{(1)}, u_1^{(2)}$ from (4.7); u_{p2}, v_{p2} - values with approximation $O\left(h^2\right)$ from (4.8); u_{p4} v_{p4} - values with approximation $O\left(h^4\right)$ from (4.9); u_{p8} v_{p8} - values with $O\left(h^8\right)$ from (4.11); u_{t2}, v_{t2} - values with $O\left(h^2\right)$ from (4.12) (method of lines); u_{t4}, v_{t4} - values with $O\left(h^4\right)$ from (4.11) (method of lines).

Table 1

The values of $u(0.5,t)$ in order of time

t	u_*	v_*	u_{p8}	v_{p8}	u_{p4}	v_{p4}
0.1	.372708	.740556	.372696	.740546	.383	.750
0.2	.138911	.413111	.138902	.413092	.147	.482
0.3	.051773	.205068	.051768	.205052	.056	.218
0.4	.019296	.095475	.019294	.095464	.021	.104
0.5	.007192	.042682	.007191	.042676	.008	.048

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Table 2

The values of $u(0.5,t)$ in order of time

t	u_{p2}	v_{p2}	u_{t2}	v_{t2}	u_{t4}	v_{t4}
0.1	.301	.663	.449	.809	.366	.737
0.2	.091	.308	.202	.525	.133	.402
0.3	.027	.126	.091	.308	.048	.195
0.4	.008	.048	.041	.171	.017	.088
0.5	.002	.017	.018	.092	.006	.038

Application example2

In [1] by modelling textile package are considered the equations for diffusion of heat and moisture in the following form

$$\begin{cases} a_1\partial C/\partial t - b_1\partial T/\partial t = c_1\partial^2 C/\partial x^2 \\ -b_2\partial C/\partial t + a_2\partial T/\partial t = c_2\partial^2 T/\partial x^2 \end{cases}, \tag{4.13}$$

where $a_1 = 1+\gamma\sigma$, $a_2 = 1+\varepsilon\varpi$, $b_1 = \gamma\varpi$, $b_2 = \varepsilon\sigma$, $c_1 = gD_A$, $c_2 = K/(c\rho)$, $\gamma = (1-v)/v\rho_S$, $\varepsilon = q/c$, $M = const + \sigma C - \varpi T$ - the amount of moisture absorbed by unit mass of fibre, T - the temperature, C - concentration of water vapour in the air space, σ, ϖ - constants, D_A - diffusion coefficient for moisture on air, v - fraction of the total volume of the package is occupied by air and 1 – v by fibre of density ρ_S, g - factor of fibres orientation, c - the specific heat of the fibres, K - the heat conductivity of the package, ρ - the density of the package, q - the heat evolved when the water vapour is absorbed by the fibres.

The system of two PDE – s (4.13) is in form (1.1), where

$Q = 0$, $u = (C,T)^T$ - vector – column, $G = \begin{pmatrix} a_1 & -b_1 \\ -b_2 & a_2 \end{pmatrix}$, $L = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$, $c_1 > 0$, $c_2 > 0$,

$\det(G) = a_1a_2 - b_1b_2 = (1+\gamma\sigma)(1+\varepsilon\varpi) - \gamma\varpi\varepsilon\sigma = 1+\gamma\sigma+\varepsilon\varpi > 0$.

The boundary conditions in the element of textile package are in the form $(x = l)$

$$\begin{cases} L\dfrac{\partial u(l,t)}{\partial x} = -\alpha_l\left(-u(l,t) + U_l\right) \\ u(0,t) = U_0 \end{cases}, \tag{4.14}$$

where $U_0 = (C_0, T_0)^T$, $U_l = (C_l, T_l)^T$ are given vectors – column, α_l - diagonal – matrix $\begin{pmatrix} \alpha_{lC} & 0 \\ 0 & \alpha_{lT} \end{pmatrix}$, where α_{lC}, α_{lT} - corresponding transfer coefficients.

The 3 – point finite – difference scheme is (3.1) and following system of ODE – s is in form (3.7), where $L^{-1} = \begin{pmatrix} c_1^{-1} & 0 \\ 0 & c_2^{-1} \end{pmatrix}$.

The vector $\varphi(x) = (C_*(x), T_*(x))^T$ is the initial distributions of C and T in the package by $t = 0$.

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Conclusions

The aim of this paper was to continue and improve the methods described in [2-2-5-8]. The approximations of initial – boundary value problem of the system of the partial differential equations (PDE) to initial value problem for a system of ordinary differential equations (ODE) of the first or second order is considered. The described method is based on the finite – volume method. It is possible to solve transfer problems of $m > 2$ different substances (concentration, heat, moisture, and e. c.) in plate due to the obtained finite – difference vector scheme. The computations were processed by mathematical system MAPLE. The accuracy of obtained method was verified in the example 1 (due to the exact solution of PDE (4.7)). The most precise results are achieved from (4.11) with the error $O(h^8)$: 3 –4 signs of accuracy (Table 1) and from (4.11) (method of lines) with the error $O(h^4)$: 1- 2 signs of accuracy (Table 2).

Taking advantages of computer technique and appropriate mathematical approach now it is possible to deal with such theoretical and practical problems, solutions of them in the past was impossible (example 2).

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