Application Of The Interlaced Sweep Method For The Solution Of Problems In Field Theory

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Abstract. **For solution of problems in field theory the method of sweep is very popular. The authors suggest a very effective method of interlaced sweep. The essence of the interlaced sweep method lies in the fact that matrix of the linear algebraic equations system is broken into parts and the solution of separate blocks is sought by direct methods. Usually for each line of the grid a separate block is created. The system of block equations has a tridiagonal matrix where only elements of the main diagonal and two neighboring diagonals are different from zero. The system of equations with such a matrix is easily solved by the method of elimination of unknowns.**

   By solving the problems by the finite element method, the nodes of touching of neighboring elements can be placed on curved lines, and the sweep on these lines can be performed observing the principle of interlaced sweep. By following this method, the neighboring lines should not belong to the same half-step.

   Keywords: **sweep method, differential equation, system of equations.**

I INTRODUCTION

The solution of wide range of practically important problems can be reduced to calculation of various fields which are described by differential equations in partial derivatives [1]. By application of the finite difference method or of the finite element method, the solution of differential equation in partial derivatives is reduced to solution of system of equations. The matrix of this system usually has very high order where only small proportion of coefficients differs from zero [2].

The solution of this system by the method of elimination of unknown according to Gauss scheme is inefficient because it leads to increase of number of unequal to zero coefficients in the matrix.

For solution of difference equations problems in field theory the method of sweep is very popular. There are many different modifications of this method. The speed of convergence of different sweep method modifications is influenced by relative value of the matrix's diagonal elements relating to the previous step, to alternation of sweep directions, to type of displacement used, to periodic change of displacement for reverse and other factors [3]. The authors suggest a very effective method of interlaced sweep which is neither a method of simultaneous displacements, nor a method of successive displacements along the lines.

II DESCRIPTION OF THE INTERLACED SWEEP METHOD

The essence of the sweep method lies in the fact that matrix of the linear algebraic equations system received as a result of the finite difference approximation of differential equation in partial derivatives, is broken into parts and the solution of separate blocks is sought by direct methods. Moreover, the equations included in the block are met exactly. Usually for each line of the grid a separate block is created. The equations system of block has a tridiagonal matrix where only elements of the main diagonal and two neighboring diagonals are different from zero. All other elements of this matrix are equal to zero. The system of equations with such a matrix is usually solved by the method of elimination of unknowns.

In the case of interlace sweep application in two-dimensional area, each iteration consists of two half steps which alternate:

- sweep on odd gridlines;
- sweep on even gridlines.

Let us denote:

k – number of grid line;

n – number of grid node on the line;

U_{kn} – component of the required vector of the grid node with indices k and n;

PX_{kn} – coefficient of the matrix corresponding to the grid node with indices k and n of the line directed along X - axis;

PY_{kn} – coefficient of the matrix corresponding to the grid node with indices k and n of the line directed along Y - axis;

F_{kn} – right-hand member of the equation which corresponds to the grid node with indices k and n;

j – serial number of iteration;

ni – current iteration number;

kr – relaxation coefficient;

h – step of change of grid nodes coordinates.

When solving problems for one-dimensional field by the sweep method, all diagonal elements of the matrix correspond entirely to the current step, i.e. relaxation coefficient is equal to 1.
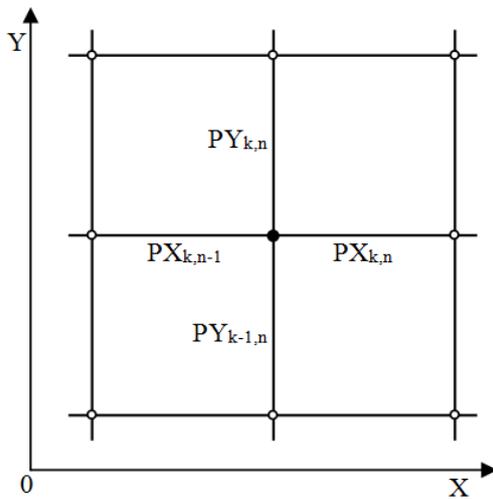

Fig. 1 Designations of grid elements.

To compensate partially the errors resulting from the transfer of some matrix coefficients of the initial equation system to the previous half-step and the increase of the convergence speed in solving the two-dimensional and three-dimensional problems, the recurrence formulas of iterative process can be changed so that the components of the required vector are determined by a factor of relaxation directly inside the block. In other words, each component of the required vector would be calculated based on the assumption that all other components of the vector have been calculated with the same coefficient of relaxation.

To achieve this goal, the recurrence formula of the half-step of the iteration process for n - the grid node depicted in Fig. 1 and sweep along the grid line directed along the x-axis must be written in the form (1).

$$PX_{k,n-1} \cdot U_{k,n-1}^{j+1/2} - DX_{kn} \cdot U_{kn}^{j+1/2} + PX_{kn} \cdot U_{k,n+1}^{j+1/2} =$$
$$= F_{kn} - PY_{k-1,n} \cdot U_{k-1,n}^{j} - PY_{kn} \cdot U_{k+1,n}^{j} +$$
$$+ \left(1 - \frac{1}{kr}\right) \cdot \left(PX_{k,n-1} + PX_{kn}\right) \cdot U_{kn}^{j-1/2}$$

where:

$$DX_{kn} = \left(PX_{k,n-1} + PX_{kn}\right)/kr + PY_{k-1,n} + PY_{kn} \qquad (1)$$

From the above recurrence formula of the iterative process follows that the coefficients of the main diagonal of the matrix in one-dimensional area are substituted by corresponding coefficients of the main diagonal of the matrix for initial problem for two or three dimensional area. Therefore, only that part of diagonal elements should be divided be the coefficient of relaxation which corresponds to the diagonal elements of the matrix of one-dimensional area. In the case of a sweep along the grid line which is directed along the y-axis, the coefficients of the main diagonal of the matrix of one dimensional area are equal to the sum of coefficients $PY_{k-1,n}$ and $PY_{k,n}$ with the opposite sign.

One part of the diagonal element of the matrix of one-dimensional area, obtained as a result of division by the coefficient of relaxation refers to the current half-step, but the remaining part refers to previous iteration. If the coefficient of relaxation is equal to one then all diagonal elements of the matrix of the initial problem relate entirely to the current half-step.

III DESCRIPTION OF CONTROL EXAMPLES

To evaluate the effectiveness of interlaced sweep method in solving two-dimensional problems the following equation (2) was being solved in the square area with dimensions 32×32, containing 900 internal grid nodes.

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma \frac{\partial U}{\partial y}\right) = f \qquad (2)$$

This equation has exact analytical solution (3):

$$U(x,y) = x^3 y^3 \qquad (3)$$

by

$$\sigma = \frac{1}{xy}, \qquad f = 3\left(x^2 + y^2\right) \qquad (4)$$

The potentials in the boundary nodes of the grid must be calculated by the function $U(x,y) = x^3 y^3$. To solve this equation the following program was written:

```
#include "stdafx.h"
#include<stdio.h>
#include <conio.h>
#include<math.h>

double u[32][32], px[32][32], py[32][32];
double d[32], b[32], f[32][32];
double x, y, h, maxr, c, kr, km;
int i, j, m, k, n, ni;

void iterx(void);
int _tmain(int argc, _TCHAR* argv[])
{
  kr = 1.74;
  km =  1.0 - 1.0 / kr;

  // calculation of the parameters
  // of the grid
  h = 0.1;
  y = 1.0;
  for(i = 0; i < 32 ; i++ ){
    x = 1.0;
    for(j = 0; j < 32; j++ ){
      u[i][j] = x * x * x * y * y * y;
      px[i][j] = 1.0 / (y * (x + h / 2.0));
      py[i][j] = 1.0 / (x * (y + h / 2.0));
      f[i][j] = 3 * h * h * (x * x + y * y);
      x = x + h;
    }
    y = y + h;
  };

  // assign 0.0 to the potentials of
  // all internal nodes of the grid
  for(i = 1; i < 31; i++)
    for(j = 1; j < 31; j++)
      u[i][j] = 0.0;

  // solution of the problem
  ni = 0;
  do {
    maxr = 0;
    // sweep on odd gridlines
    for(k = 1; k < 31; k += 2) iterx();
    // sweep on even gridlines
    for(k = 2; k < 31; k += 2) iterx();
    ni++;
    // output of the results
    printf("\n%+15.5e", maxr);
    // _getch();
  }while(maxr >  0.005);
  printf("\n%+6.3f%+8d", kr, ni);
  printf("\n%+16.5e%+16.5e%+16.5e",
    u[10][10], u[15][15], u[20][20]);
  _getch();
  return 0;
}
```

```
// sweep function along X axis
void iterx(void){
// calculation of diagonal coefficients
// and free members
  for(n = 1; n < 31; n++){
    d[n] = - (px[k][n-1] + px[k][n]) / kr -
            py[k-1][n] - py[k][n];
    b[n] = f[k][n] - py[k-1][n]*u[k-1][n] -
      py[k][n] * u[k+1][n] +
      km * (px[k][n-1] + px[k][n])*u[k][n];
  }

  // correction of free members at the
  // beginning and at the end of the line
  b[1] = b[1] - px[k][0] * u[k][0];
  b[30] = b[30] - px[k][30] * u[k][31];
  // direct sweep
  for(n = 2; n < 31; n ++){
    d[n] = d[n]-px[k][n-1]*px[k][n-1]/d[n-1];
    b[n] = b[n]-b[n-1]*px[k][n-1]/d[n-1];
  }

  // reverse sweep
  c = b[30] / d[30];
  if(fabs(u[k][30] - c) > fabs(maxr))
      maxr = fabs(u[k][30] - c);
  u[k][30] = c;
  for(n = 29; n > 0; n --){
    b[n] = b[n] - px[k][n] * u[k][n + 1];
    c = b[n] / d[n];
    if(fabs(u[k][n] - c) > fabs(maxr))
      maxr = fabs(u[k][n] - c);
    u[k][n] = c;
  }
}
```

By solving the problem using the Microsoft Visual Studio Express Edition 2012, the absolute value of maximum correction to the solution has decreased in 10^6 times in 47 iterations by kr=1.74. The total error of the finite approximation and other factors in the middle point of the area with coordinates (2.5, 2.5) is equal to 0.016%.

To evaluate the effectiveness of the interlaced sweep method in solving three-dimensional problems the following equation (5) was being solved in the area having the form of the cube with dimensions 32×32×32, containing 27000 internal grid nodes.

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\sigma\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(\sigma\frac{\partial U}{\partial z}\right) = f \qquad (5)$$

This equation has an exact analytical solution (6):

$$U(x,y,z) = x^3 y^3 z^3 \qquad (6)$$

at

$$\sigma = \frac{1}{xyz}, \qquad f = 3\left(x^2 y^2 + x^2 z^2 + y^2 z^2\right) \qquad (7)$$

The potentials in the boundary nodes of the grid must be calculated by the function $U(x,y,z) = x^3 y^3 z^3$.

To solve the problems for three-dimensional area the principle of interlaced sweep must be observed in all planes which have sweep lines.

When solving the problems the absolute value of maximum correction for solution is reduced in 10^6 times in 47 iterations by kr=7.9. The total error of the finite approximation and other factors in the midpoint of the field with coordinates (2.5, 2.5, 2.5) were equal to 0.017%.

Relatively large value of the coefficient of relaxation is explained by the fact that in three-dimensional case just four coefficients of the initial system matrix relate to the previous half-step and only small part of the coefficient which belongs to the main matrix diagonal in the initial system is divided by the coefficient of relaxation.

To solve the problems in two and three dimensional fields, the authors decided to modify the sweep method for sequential displacement along the grid lines by means of substitution of recurrence formulas of the iteration process for the sweep method by replacement formulas for two and three dimensional area where only one part of the diagonal element corresponding to the matrix of the one-dimensional field is divided by the coefficient of relaxation. This modification of the sweep method has increased the speed of convergence of iteration process in two and three dimensional fields in several times.

The execution of iterations by solving the control example by all above mentioned methods was interrupted when the maximum module of correction to the solution decreased in 10^6 times.
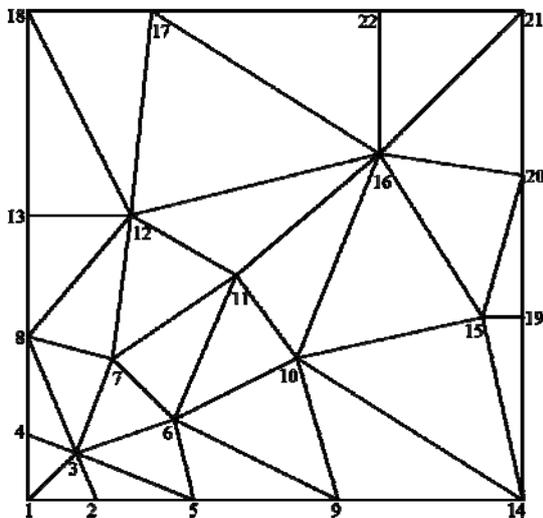
The results of the comparison of the interlaced sweep method with such methods as: the sweep method for successive displacement along the grid lines, the modified sweep method, and the method of successive over-relaxation for two-dimensional area are shown in the table 1.

Similar results for the three-dimensional area are shown in the table 2.

TABLE 1
THE RESULTS OF SOLVING THE CONTROL EXAMPLE IN TWO-DIMENSIONAL AREA

Name of the method	Coefficient of relaxation	The number of iteration	error
Successive over-relaxation	1.77	98	0.28%
Sweep for successive displacement along the lines	1.0	368	0.06%
Modified sweep method	1.76	54	0.01%
Interlaced sweep	1.74	47	0.016%

TABLE 2
THE RESULTS OF SOLVING THE CONTROL EXAMPLE IN THREE-DIMENSIONAL AREA

Name of the method	Coefficient of relaxation	The number of iteration	error
Successive over-relaxation	1.81	95	0.006%
Sweep for successive displacement along the lines	1.0	416	0.05%
Modified sweep method	7.9	65	0.012%
Interlaced sweep	7.9	47	0.017%

IV CONCLUSION

The number of operations to be performed at each point of the grid when solving the problems by the methods listed in the table differs slightly, and the speeds of convergence differ in several times. The modification of the sweep method for sequential displacement along the grid lines has increased the speed of its convergence in 6 times, but the method of interlaced sweep turned out to be more effective. This proves the high efficiency of the interlaced sweep method.

By solving the problems by the finite element method, the nodes of touching of neighboring elements (Fig. 2) can be placed on curved lines, and the sweep on these lines can be performed observing the principle of interlaced sweep. By following this method, the neighboring lines should not belong to the same half-step.

Fig. 2 Triangular grid.

V REFERENCES

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